Worksheet: Sum and Product of Roots (Viète's Theorem)

Student Name: _____ **Date:** _____

Learning Objective:

Consolidate the understanding of **Viète's Theorem**: the relationships between the coefficients of a polynomial and the sums/products of its roots..

Question 1. Quadratic Basics

For each quadratic polynomial below, determine

(i) the sum of its roots, and
(ii) the product of its roots, **without** actually finding the individual roots.

a) $2x^2 - 3x + 4$

b) $x^2 + 6x + 5$

c) $-4x^2 + 7x - 2$

Question 2. From Root Sums to Constructing Quadratics

(a) Find a **monic** quadratic polynomial $x^2 + px + q$ if its two roots have:

- Sum of roots = 8
- Product of roots = 15

(b) Check whether your polynomial from (a) has integer solutions (you may factor or use the discriminant).

Question 3. Mixed Quadratic Practice

Consider the polynomial $3x^2 - kx + 10$. Suppose it has two (real or complex) roots r_1, r_2.

a) State $r_1 + r_2$ and $r_1 r_2$ in terms of k.

b) If you also know $r_1 + r_2 = -2$, find the value(s) of k.

c) For each possible k, determine $r_1 r_2$.

d) **Extension**: Investigate whether the roots are real or complex for each k.

Question 4. Cubic Polynomials: Viète's Formulas

A cubic polynomial has the form $p(x) = ax^3 + bx^2 + cx + d$, with roots α, β, γ.

Recall that: $\alpha + \beta + \gamma = -\frac{b}{a}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$, $\alpha\beta\gamma = -\frac{d}{a}$. .

(a) Given $p(x) = 2x^3 - 5x^2 + 4x - 3$, write down:

i. $\alpha + \beta + \gamma$

ii. $\alpha\beta + \beta\gamma + \gamma\alpha$

iii. $\alpha\beta\gamma$

(b) Without solving for α, β, γ, which might be complicated, discuss briefly how you can still know the sign of αβγ. (*Hint: look at the sign of* $-\frac{d}{a}$).

Question 5. Constructing a Cubic from Partial Root Information

A cubic $f(x) = ax^3 + bx^2 + cx + d$ has roots r_1, r_2, r_3 . You know:

- $r_1 + r_2 + r_3 = 2$
- $r_1r_2 + r_1r_3 + r_2r_3 = 1$
- $r_1 = -1$

(a) Use the fact $r_1 = -1$ to express $(r_2 + r_3)$ and (r_2r_3) in simpler terms.

(b) Write down a factor for the root $r_1 = -1$.

(c) Suppose $a = 1$ (monic). Using Viète's Theorem, find b, c, d. Hence write your cubic explicitly.

(d) (Extension) If $a \neq 1$, how would that change b, c, d?

Question 6. Quartic Sums and Products

Let the quartic polynomial be: $g(x) = ax^4 + bx^3 + cx^2 + dx + e$, with roots r_1, r_2, r_3, r_4.

Viète's Theorem for a quartic tells us:

$r_1 + r_2 + r_3 + r_4 = -\frac{b}{a}$,

$r_1r_2 + r_1r_3 + \cdots + r_3r_4 = \frac{c}{a}$,

$r_1r_2r_3 + \ldots = -\frac{d}{a}$,

$r_1r_2r_3r_4 = (-1)^4\frac{e}{a}$.

Given:

1. $r_1 + r_2 + r_3 + r_4 = 1$.
2. $r_1r_2 + r_1r_3 + \cdots + r_3r_4 = 5$.
3. $r_1r_2r_3r_4 = -6$.

(a) Write relationships for b, c, e in terms of a.

(b) Which piece of information is missing if we wanted to fully specify d?

(c) Write the quartic polynomial in the form $ax^4 + bx^3 + cx^2 + dx + e$, leaving d as an unspecified parameter.

Question 7. Higher Degree: Quick Check

Consider

$$h(x) = 2x^5 + 3x^4 - x^3 + 5x^2 + 8x - 10.$$

Let its roots be s_1, s_2, s_3, s_4, s_5. Using Viète's:

1. Find $s_1 + s_2 + s_3 + s_4 + s_5$.
2. Find $s_1\, s_2\, s_3\, s_4\, s_5$.
3. If you only needed the sum of the pairwise products $s_1\, s_2 + s_1\, s_3 + \cdots + s_4\, s_5$, which coefficient would you look at, and how do you compute it from the polynomial?

Question 8. Reciprocals of Roots

If a polynomial has nonzero roots r_1, r_2 (for a quadratic) or $r_1,\ldots, r_n$ (for degree n), we can explore sums and products of **reciprocals**:

$$\frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_n}.$$

- For a **quadratic** $ax^2 + bx + c$ with roots r_1, r_2, show that
$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2} = \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c}$$
- State a corresponding formula for the product $\frac{1}{r_1} \cdot \frac{1}{r_2}$.
- Why does the polynomial $cx^2 + bx + a = 0$ connect naturally to the reciprocals of the original polynomial's roots?

Tips for Students

- **No Need to Solve**: Often, Viète's is designed so you can find sum/product relationships without needing the actual roots.
- **Signs:** Pay attention to the sign changes. For an n-degree polynomial $a_n x^n + \cdots + a_0$, the product of all roots is $(-1)^n \frac{a_0}{a_n}$.
- **Constructing Polynomials**: If you know certain symmetric sums of roots (e.g., sum, sum of products, product, etc.), you can build the polynomial by matching those sums to the correct coefficients.
- **Complex Roots**: Even if some or all roots are complex, the same formulas hold, thanks to the general nature of polynomial factorization in the complex field.

Instructions for Submission:

- Solve the questions on a separate sheet or digitally and submit them by the given deadline.
- Justify all answers with proper reasoning.

End of Worksheet

Solutions to Sum and Product of Roots (Viète's Theorem) Worksheet

Question 1. Quadratic Basics

For each quadratic ax2+bx+cax^2 + bx + cax2+bx+c, the sum of roots is −ba-\frac{b}{a}−ab and the product of the roots is ca\frac{c}{a}ac, by Viète's theorem.

a) $2x^2 - 3x + 4$

i. Sum of roots: $-\frac{-3}{2} = \frac{3}{2}$.

ii. Product of roots: $\frac{4}{2} = 2$.

b) $x^2 + 6x + 5$

i. Sum of roots: $-\frac{6}{1} = -6$.

ii. Product of roots: $\frac{5}{1} = 5$.

c) $-4x^2 + 7x - 2$

i. Sum of roots: $-\frac{7}{-4} = \frac{7}{4}$.

ii. Product of roots: $\frac{-2}{-4} = \frac{1}{2}$.

Question 2. From Root Sums to Constructing Quadratics

(a) Find a monic quadratic if:

- Sum of roots = 8
- Product of roots = 15

If the polynomial is $x^2 + px + q$, then from Viète:

Sum = −p = 8 ⟹ p = −8,

Product = q = 15.

Hence the polynomial is:

$x^2 - 8x + 15$.

(b) Factor or check its roots:

$x^2 - 8x + 15 = (x - 3)(x - 5)$.

Indeed, the roots are 3 and 5, whose sum is 8 and product is 15. They are integers.

Question 3. Mixed Quadratic Practice: $3x^2 - kx + 10$

Let the roots be r_1, r_2. By Viète:

1. $r_1 + r_2 = -\frac{-k}{3} = \frac{k}{3}$.
2. $r_1 r_2 = \frac{10}{3}$.

(1) State in terms of k:

- Sum: $r_1 + r_2 = \frac{k}{3}$..
- Product: $r_1 r_2 = \frac{10}{3}$.

(2) If $r_1 + r_2 = -2$, then:

$\frac{k}{3} = -2 \Longrightarrow k = -6$.

(3) For $k = -6$:

$r_1 r_2 = \frac{10}{3}$.

So that remains $\frac{10}{3}$.

(4) Real or complex?

- The discriminant is $\Delta = (-k)^2 - 4 \cdot 3 \cdot 10 = k^2 - 120$.
- If $k = -6$, then $\Delta = (-6)2 - 120 = 36 - 120 = -84 < 0$.
  So the roots are **complex** for $k = -6$.

(If the question wanted us to consider other k values, we'd see when $k2 - 120 \geq 0$.)

Question 4. Cubic Polynomials: Viète's Formulas

(a) Polynomial $p(x) = 2x^3 - 5x^2 + 4x - 3$.

Let roots be α, β, γ. Then:

- $\alpha + \beta + \gamma = -\frac{-5}{2} = \frac{5}{2}$.
- $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{4}{2} = 2$.
- $\alpha\beta\gamma = -\frac{-3}{2} = \frac{3}{2}$.

(b) Sign of $\alpha\beta\gamma$:

- $\alpha\beta\gamma = \frac{3}{2}$, which is positive. So, even without solving, we know the product of the three roots is +1.5.

Question 5. Constructing a Cubic from Partial Root Information

We have a cubic $f(x) = ax3 + bx2 + cx + d$ with roots r_1, r_2, r_3 . Given:

1. $r_1 + r_2 + r_3 = 2$
2. $r_1 r_2 + r_1 r_3 + r_2 r_3 = 1$
3. $r_1 = -1$

(a) Express $(r_2 + r_3)(r_2 r_3)$

1. From $r_1 + r_2 + r_3 = 2$ and $r_1 = -1$, we get

$$-1 + r_2 + r_3 = 2 \Longrightarrow r_2 + r_3 = 3.$$

2. From $r_1 r_2 + r_1 r_3 + r_2 r_3 = 1$ and $r_1 = -1$, we get

$$(-1)r_2 + (-1)r_3 + r_2 r_3 = 1 \Longrightarrow -(r_2 + r_3) + r_2 r_3 = 1.$$

Substitute $r_2 + r_3 = 3$ into that:

$-3 + r_2 r_3 = 1 \Longrightarrow r_2 r_3 = 4.$

(b) Factor for the root $r_1 = -1$:

A root at -1implies $(x - (-1)) = (x + 1)$ is a factor.

(c) Suppose $a = 1$ (monic). Find b, c, d.

For a monic cubic $x^3 + bx^2 + cx + d$ with roots r_1, r_2, r_3:

- $r_1 + r_2 + r_3 = -b.$
- $r_1 r_2 + r_2 r_3 + r_3 r_1 = c.$
- $r_1 r_2 r_3 = -d.$

Given:

- $r_1 + r_2 + r_3 = 2$ so $-b = 2 \Longrightarrow$ b = -2.
- $r_1 r_2 + r_2 r_3 + r_3 r_1 = 1$ so $c = 1$.
- $r_1 r_2 r_3 = -1 \cdot r_2 r_3 = -1 \cdot 4 = -4 \Longrightarrow$ d = 4.

Hence the cubic is: $x^3 - 2x^2 + x + 4$.

(Optionally check (x+1)(x+1)(x+1) is indeed a factor.)

(d) If $a \neq 1$,

- We'd write a $\neq$ 1. Then from Viète:

$r_1 + r_2 + r_3 = -\frac{b}{a}, r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{c}{a}, r_1 r_2 r_3 = -\frac{d}{a}$.

We'd match those with 2, 1, and $(-1) \cdot r_2 r_3$. This leads to a consistent ratio among $\frac{b}{a}, \frac{c}{a}, \frac{d}{a},$, so the polynomial's coefficients would be scaled by some factor a.

Question 6. Quartic Sums and Products

Given a quartic $g(x) = ax^4 + bx^3 + cx^2 + dx + e$, with roots r_1, r_2, r_3, r_4:

1. $r_1 + r_2 + r_3 + r_4 = 1$.
2. $\sum_{i<j} r_i r_j = 5$.
3. $r_1 r_2 r_3 r_4 = -6$.

By Viète's:

- $r_1 + \cdots + r_4 = -\frac{b}{a} = 1 \Longrightarrow b = -a$.
- $\sum_{i<j} r_i r_j = \frac{c}{a} = 5 \Longrightarrow c = 5a$.
- $r_1 r_2 r_3 r_4 = \frac{e}{a}(-1)^4 = \frac{e}{a} = -6 \Longrightarrow e = -6a$.

So:

(a) $b = -a,\ c = 5a,\ e = -6a$.

(b) Missing is the sum of triple products $\sum_{i<j<k} r_i r_j r_k$, which is $-\frac{d}{a}$. So we don't have enough info to solve for d.

(c) The quartic polynomial is:

$ax^4 - ax^3 + 5ax^2 + dx - 6a$,

with d free. (We only know it's $-a\sum_{i<j<k} r_i r_j r_k$)

Question 7. Higher Degree: Quick Check

$h(x) = 2x^5 + 3x^4 - x^3 + 5x^2 + 8x - 10$, roots $s_1, \ldots, s_5$. By Viète:

1. **Sum of roots**: $s_1 + s_2 + s_3 + s_4 + s_5 = -\frac{3}{2}$.
2. **Product of roots**: $s_1 s_2 s_3 s_4 s_5 = (-1)^5 \cdot \frac{-10}{2} = -1 \cdot (-5) = 5$.
3. **Sum of pairwise products**: $\sum_{i<j} r_i r_j = \frac{coefficient\ of\ x^3}{\text{leading coeff}}$ *with sign carefully accounted for*. Actually, for a 5th-degree:

$$2x^5 + \underbrace{3x^4}_{a_4} + \underbrace{-x^3}_{a_3} + \ldots$$

The sum of pairwise products is the coefficient of x^3 **divided by** the leading coefficient, but with the sign pattern:

- The general pattern for a 5th degree is:

$s_1 + s_2 + \cdots + s_5 = -\frac{a_4}{a_5}$,

$\sum_{i<j} s_i s_j = \frac{a_3}{a_5}$, …

so indeed $\sum_{i<j} s_i s_j = \frac{-1}{2} = -\frac{1}{2}$.

*(In general, for an n-degree polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots$, the sign alternates:

- sum of roots = $-\frac{a_{n-1}}{a_n}$,
- sum of pairwise products = $\frac{a_{n-2}}{a_n}$,
- sum of triple products = $-\frac{a_{n-3}}{a_n}$, etc. So here $n = 5$, the coefficient for x^3 is $a_3 = -1$, which gets a **positive** sign in the formula, i.e. $\frac{a_3}{a_5}$.)*

Thus: $\sum_{i<j} s_i s_j = \frac{-1}{2}$.

Question 8. Reciprocals of Roots

Quadratic $ax^2 + bx + c = 0$ with roots $r_1, r_2 \neq 0$. Then:

1. $r_1 + r_2 = -\frac{b}{a}$, $r_1 r_2 = \frac{c}{a}$.
2. Sum of reciprocals:

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2} = \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c}.$$

3. Product of reciprocals:

$$\frac{1}{r_1} \cdot \frac{1}{r_2} = \frac{1}{r_1 r_2} = \frac{1}{\frac{c}{a}} = \frac{a}{c}.$$

4. Polynomial for reciprocals:
   - If r_1, r_2 are nonzero, then the equation satisfied by $\frac{1}{r_1}, \frac{1}{r_2}$ can be found by substituting $x \to \frac{1}{x}$ into $ax2 + bx + c = 0$, or by reversing coefficients:

$cx^2 + bx + a = 0$

has solutions $\frac{1}{r_1}, \frac{1}{r_2}$.

Final Remarks

- These solutions illustrate how **sum and product** (and higher-order symmetric sums) can be read directly from the polynomial coefficients using **Viète's Theorem**.
- No need to explicitly solve for roots in many cases—**that's the beauty** of these relationships.
- Students should double-check signs carefully, especially for higher-degree polynomials (note the alternating pattern).