Lesson Plan: Sum and Product of Roots (Viète's Theorem)

Subject: Mathematics
Course: IB Mathematics Analysis and Approaches
Level: IB HL
Topic: Sum and Product of the Roots of Polynomial Equations
Duration: 80 minutes
Prerequisite Knowledge:

- Basic understanding of polynomials and their factor forms.
- Familiarity with Factor Theorem and the roots of quadratic equations.

Learning Objective:

By the end of this lesson, students will be able to:

1. **Recall** Viète's relations for quadratic, cubic, and higher-degree polynomials with real (or complex) coefficients.
2. **Derive** the sum and product (and other symmetric sums) of the roots from the expanded polynomial coefficients.
3. **Apply** these relations to find unknown coefficients given certain information about sums or products of roots, or to construct polynomials from known roots.
4. **Extend** the concept to identify how these patterns scale up to quartic, quintic, and general n-th degree polynomials.

Resources & Materials

- **PowerPoint slides**: "AAH2_12_7 Sum and product of the roots of polynomial equations.pptx."
- **Whiteboard / interactive board** for note-taking and illustrating expansions.
- **Student handouts** (optional) with guiding questions and practice problems.
- **Graphing calculators** or other CAS (optional for verification).

---

1. Lesson Introduction (10 minutes)

Engagement – Inquiry Prompt:

Hook / Brainstorm: Ask students:

"If a quadratic $ax^2 + bx + c = 0$ has solutions r_1, r_2, can you guess a relationship between $r_1 + r_2$ and the coefficients? What about $r_1 r_2$?"
Let them brainstorm in small groups or pairs. Some may recall that $r_1 + r_2 = -\frac{b}{a}$ and $r_1 r_2 = \frac{c}{a}$

Recap: Remind them of the standard form $ax^2 + bx + c = 0$ and the known factorizations $(x - r_1)(x - r_2) = 0$.

---

2. Exploration – Guided Inquiry (20 minutes)

Student-Led Derivation for Quadratics

- Have students write $(x - r_1)(x - r_2) = x2 - (r_1 + r_2)x + r_1\, r_2$.
- Compare term-by-term with $ax^2 + bx + c$.
- Ask them to match coefficients:

$a = 1$(if monic),$-(r_1 + r_2) = b/a$, $r_1\, r_2 = c/a$.

This direct exploration helps them realize "sum of roots = $-b/a$" and "product of roots = c/a."

Transition to Cubics

- Show the slides or write on the board: If a cubic polynomial $ax^3 + bx^2 + cx + d$ factors as $(x - x_1)(x - x_2)(x - x_3)$, how do we get the sums $x_1 + x_2 + x_3$, etc.?
- Let students try to expand $(x - x_1)(x - x_2)(x - x_3)$ at least partially and compare to $ax^3 + bx^2 + cx + d$.
- Identify the "sum of roots," the "sum of product of pairs," and the "product of all three" from the final expanded form.

---

3. Explanation – (25 minutes)

Viète's Theorem:

- **Slides:** Present the general statements from the PowerPoint for quadratics, cubics, and the extension to quartic/higher degrees.
- **Emphasize patterns:** For an n-degree polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with roots $r_1, r_2, \ldots, r_n$,

$$r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}\,, \quad r_1\ r_2\ \cdots\ r_n\ = \frac{a_0}{a_n}\,.$$

And so on for the sums of products of roots taken 2 at a time, 3 at a time, etc.

Small Group Exercise:

- Provide a polynomial (e.g., a cubic or quartic). Ask them to identify the sum of its roots, the sum of the products of the roots taken two at a time, and the product of all roots, without actually solving for the roots.
- They compare answers with each other, verifying they have used the correct coefficient–root relationships.

Elaborate (15 minutes)

1. **Constructing Polynomials from Known Root Conditions**

- **Example:** If the sum of the roots is 5 and the product of the roots is 6 for a monic quadratic, what is the polynomial? Encourage them to say $x^2 - (5)x + 6$.
- Use the slides that show how to construct a polynomial if roots are $3 + 2i$ and $3 - 2i$, or to handle a scenario like: "Find a cubic with integer coefficients if it has roots at -1, 2, and sum of roots is 4 (implying a missing third root)."

2. **Inquiry Problem:**
   - *"Can we find a polynomial of degree 4 if we know the sum of its roots is 1, the sum of the product of its roots taken two at a time is 5, and the product of all roots is -6?"*
   - Let students outline or do the partial factor expansions. Possibly discuss the sign changes due to the term $(-1)^n$.

E. Evaluate / Reflect (5 minutes)

- **Check for Understanding:** Ask quick prompts:
  - "What is the sum of the roots of $2x^5 + 5x^4 - x + 6$?" (Answer: $-\frac{5}{2}$.)
  - "If a cubic polynomial with leading coefficient 3 has roots that multiply to 10, what does that tell you about d if the polynomial is $3x^3 + bx^2 + cx + d$?" (Answer: $d = -30$, because product of roots is $-\frac{d}{3} = 10 \implies d = -30$.)

F. Possible Extensions / Homework

- **Practice Sheet:** Provide a set of polynomials (quadratic, cubic, quartic) for which students:
  1. Find the sum/product of roots (and other symmetric sums).
  2. Construct polynomials from given root information.
- **TOK Connection**: Pose a short reflection: "Why is it we can talk about sums and products of complex roots just as easily as real roots? How does this rely on the structure of arithmetic in the complex number field?"

Additional Notes for the Teacher

1. Focus on Understanding: Emphasize that students do not need to solve for individual roots to know these sums/products.
2. Encourage Conjectures: Let students propose general forms; confirm with expansions.
3. Variety: Include examples where coefficients are not just 1 or -1 but also 2, 3, or other integers, so they see the role of dividing by a_n.
4. Assessment: Short, in-class tasks or group presentations can gauge how well students grasp Viète's relationships.