# Worksheet: The Conjugate Root Theorem

**Student Name: ______________________________________ Date: _______________**

**Learning Objective:**

Apply the Conjugate Root Theorem to factorise Real coefficient polynomials.

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## Question 1. Basic Factorization with Known Complex Root

Let

$$f(x) = x^3 + 2x^2 + 4x + 8..$$

You are informed that $x = 2i$ is a root of $f$.

(a) Show that $2i$ is indeed a root by direct substitution.
(b) Use the **conjugate root theorem** to identify another root.
(c) Factorize $f(x)$ fully over the complex numbers and find the remaining root(s).

## Question 2. Determining a Parameter Given a Complex Root

Consider the polynomial

$$g(x) = x^3 + (a + 2)\,x^2 + (5 - a)\,x + 4,$$

where $a$ is a real constant. Suppose it is known that $x = i$ is a root.

1. Show how to use the condition $g(i) = 0$ to find the value of $a$.
2. Hence, factorize $g(x)$ completely.
3. State all roots clearly.

## Question 3. Conceptual / Proof-Style Question

Explain, in your own words, **why** the conjugate of any complex root must also be a root for a polynomial with real coefficients. Your explanation might:

- Reference the fact that real coefficients remain unchanged when conjugated.
- Show how $(f(z))^* = f(z^*)$ for any polynomial $f$ with real coefficients.
- Conclude that if $f(z) = 0$, then $f(z^*) = 0$.

*(A fully rigorous proof is not required; a concise argument referencing the properties of conjugation is sufficient.)*

## Question 4. Mixed Real and Complex Roots

A polynomial of degree 4,

$$p(x) = 2x^4 + bx^3 + cx^2 + dx - 30,$$

has real coefficients. You know two of its roots are $\alpha = 3$ (real) and $\beta = 2 - i$.

1. Use the fact that $\alpha=3$ is a root to state a factor of $p(x)$.
2. Apply the conjugate root theorem to determine another root directly from $\beta$.
3. Write down three factors of $p(x)$ that you can already identify.
4. Suggest a general strategy for finding the fourth root. Find the value of $b$, $c$, and $d$.

## Question 5. Quadratic Construction

You are given that a quadratic polynomial $q(x) = x^2 + px + r$ (with $p, r \in \mathbb{R}$) has one root $1 + \sqrt{3}i$.

1. State the other root, using the conjugate root theorem.
2. By using Vieta's formulas (sum and product of roots in terms of $p$ and $r$), show that

$$p = -2, r = 4.$$

3. Verify that $q(x) = x^2 - 2x + 4$ indeed has the given root.

## Question 6. Polynomial with a Double Complex Root

Let

$$h(x) = (x - 3)(x - (1 + 2i))^2(x - (1 - 2i))^2.$$

1. Expand this polynomial fully to write $h(x)$ in standard form $a_0 + a_1x + a_2x^2 + \cdots$ .
2. Notice that the repeated factors $(1 + 2i)$ and $(1 - 2i)$ appear. Explain briefly how the conjugate root theorem remains consistent if a complex root has multiplicity $> 1$.
3. Identify all roots and their multiplicities clearly.

## Question 7. Using the Conjugate Root Theorem to Simplify Polynomial Division

Suppose

$$r(x) = x^4 + 4x^3 + mx^2 + nx + 9,$$

where $m$ and $n$ are real numbers. You are told $r(x)$ has a root $\gamma = -1 + 2i$

1. Write down the conjugate root $\gamma^*$.
2. Hence state two linear factors of $r(x)$.
3. Outline how you would use polynomial division (or factor by grouping) to reduce $r(x)$ to a quadratic factor.
4. If you were given a further real root $x = 1$, how would that help you solve for $m$ and $n$? (You need not complete all algebra unless you want extra challenge.)

### Question 8. Reflective / TOK-Style Extension

In the realm of real-coefficient polynomials, the **conjugate root theorem** guarantees non-real solutions come in pairs. Discuss briefly:

- How does this connect to the Fundamental Theorem of Algebra (which states every degree $n$ polynomial has $n$ complex roots, counting multiplicities)?
- If we changed the coefficient set from real numbers to something else (e.g., complex numbers themselves, or polynomials over finite fields), would the statement of the "conjugate root theorem" necessarily hold? Provide a brief rationale without extensive formal proofs.

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### Instructions for Submission:

- Solve the questions on a separate sheet or digitally and submit them by the given deadline.
- Justify all answers with proper reasoning.

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**End of Worksheet**

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# Solutions to Fundamental Theorem of Algebra Worksheet

**Note to Educators:**

- This worksheet is designed to progressively increase in difficulty, covering different aspects of the Fundamental Theorem of Algebra.
- Some questions require the use of conjugates and polynomial factorization.
- Encourage students to use complex number properties and De Moivre's Theorem where relevant.
- The use of a calculator is allowed, but encourage exact answers when possible.

## Question 1. Basic Factorization with a Known Complex Root

### Question Recap

$f(x) = x^3 + 2x^2 + 4x + 8$, and $x = 2i$ is a root.

1. Show $f(2i) = 0$.
2. Find another root using the Conjugate Root Theorem.
3. Factorize completely over $\mathbb{C}$ and state all roots.

### Solution

1. **Verification**

$f(2i) = (2i)^3 + 2(2i)^2 + 4(2i) + 8.$

- $(2i)^2 = -4.$
- $(2i)^3 = (2i)^2 \cdot (2i) = (-4)(2i) = -8i.$

Substitute:

$f(2i) = -8i + 2(-4) + 8i + 8 = -8i - 8 + 8i + 8 = 0.$

So $2i$ is indeed a root.

2. **Another Root**
   Because the coefficients of $f$ are real, the Conjugate Root Theorem implies that $-2i$ is also a root.

3. **Factorization**

- From the two complex roots, $(x - 2i)(x + 2i) = x^2 + 4$ is a factor.
- Divide $f(x)$ by $(x^2 + 4)$ (using long division or synthetic division) to get the other factor $x + 2$.

Thus

$f(x) = (x^2 + 4)(x + 2).$

The remaining root is $x = -2$.

All roots: $x = 2i, -2i,$ and $-2$

## Question 2. Determining a Parameter Given a Complex Root

### Question Recap

$g(x) = x^3 + (a + 2)x^2 + (5 - a)x + 4$, with $a \in \mathbb{R}$. Supposedly $x = i$ is a root.

1. Show how to find $a$.
2. Factorize completely.
3. State all roots.

### Solution Discussion

1. **Evaluate $g(i)$**

$$g(i) = i^3 + (a + 2)\, i^2 + (5 - a)i + 4.$$

- $i^2 = -1$, $i^3 = i^2 \cdot i = -1 \cdot i = -i$.

  So

$g(i) = -i + (a + 2)(-1) + (5 - a)i + 4.$

Group real and imaginary parts:

- **Real part**: $-(a + 2) + 4 = -a - 2 + 4 = -a + 2$.
- **Imag part**: $(-1 + (5 - a))i = (4 - a)i$.

For $g(i)$ to be zero, we need both:

$-a + 2 = 0$ and $4 - a = 0$.

This system gives $a = 2$ from the first equation and $a = 4$ from the second, which is **contradictory**.

Therefore, there is **no real *a*** that makes $i$ a root. In other words, no real value of $a$ satisfies g(i) = 0.

2. **Conclusion**
   Since the system for the real and imaginary parts cannot be solved simultaneously by a single real $a$, there is **no factorization** in real coefficients that has $i$ as a root.

**Answer**: **No real value** of $a$ makes $x = i$ a root, so parts (2) and (3) do not apply (the premise is impossible under real $a$).

## Question 3. Conceptual / Proof-Style Question

Explain why the conjugate of any complex root must also be a root for polynomials with real coefficients.

### Key Points in a Good Explanation

- **Definition**: If a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has real coefficients $a_k \in \mathbb{R}$, and if $z = a + bi$ is a root, then $f(z) = 0$.
- **Conjugation Preserves Real Coefficients**:

- $(a_k)^* = a_k$ for each coefficient.
- $(zn)^* = (z^*)^n$.
- Conjugation distributes over sums and products:

  $(xy)^* = x^*\ y$ and $(x + y)^* = x^* + y^*$.

- **Argument**:

$f(z^*) = a_n(z^*)^n + \cdots + a_1z^* + a_0 = (a_nz^n)^* + \cdots + (a_1z)^* + a_0^* = (a_nz^n + \cdots + a_1z + a_0)^*$

$= (f(z))^* = 0^* = 0.$

- **Conclusion**: If $z$ is a root, so is $z^*$.

**Question 4. Mixed Real and Complex Roots**

**Question Recap**

A quartic polynomial $p(x) = 2x^4 + bx^3 + cx^2 + dx – 30$ (all real coefficients) has roots 3 and 2 – i.

1. State a factor from $x = 3$.
2. Use CRT to identify another root from $2 - i$.
3. Write three factors.
4. Suggest a strategy to find the fourth root. Find the value of $b$, $c$, and $d$.

**Solution Outline**

1. **Known Real Root**: $x = 3$ means $(x - 3)$ is a factor.
2. **Conjugate Root**: If $2 – i$ is a root, then $2 + i$ is also a root. So $(x - (2 - i))(x - (2 + i))$ is a factor.
3. **Three Factors**:

$(x - 3)$, $(x - (2 - i))$, $(x - (2 + i))$.

In real-coefficient form, $(x - (2 - i))(x - (2 + i)) = x^2 – 4x + 5$.

4. **Remaining Root / Factor**:
   Since $p$ is degree 4, we can do:

$p(x) = 2 \times (x - 3)\,(x^2 - 4x + 5)$ (some linear factor).

To find that linear factor (and thus the fourth root):

- Either **divide** $p(x)$ by $(x - 3)(x^2 - 4x + 5)$.
- Or use relationships among coefficients and sums/products of roots if more information about $b$, $c$, $d$ is given.

$b = –4$, $c = –12$, $d = 44$

**Question 5. Quadratic Construction**

**Question Recap**

$q(x) = x^2 + px + r$ with real $p$, $r$. One root is $1 + \sqrt{3}i$ .

1. State the other root (by CRT).
2. Use Vieta's formulas to find $p$ and $r$.
3. Verify $q(x)$.

**Solution**

1. **Conjugate Root**: $1 - \sqrt{3}i$.
2. **Sum and Product of Roots** (Vieta's relations for a monic quadratic $x^2 + px + r$):
   - Sum of roots: $(1 + \sqrt{3}i) + (1 - \sqrt{3}i) = 2$.

     By Vieta, sum of roots = $-\frac{a_{n-1}}{a_n} - p$. So $-p = 2 \Longrightarrow p = -2$.
   - Product of roots: $(1 + \sqrt{3}i)(1 - \sqrt{3}i)$

     $= 1 - (\sqrt{3}i)^2$

     $= 1 - (3 \cdot i^2)$

     $= 1 - 3(-1)$

     $= 1 + 3 = 4$.

     By Vieta, product of roots = $(-1)^n \frac{a_0}{a_n} = r$. Thus $r = 4$.
3. **Polynomial**: $q(x) = x^2 - 2x + 4$.
   A quick check: $q(1 + \sqrt{3}i) = 0$.

   $$0 = (1 + \sqrt{3}i)^2 - 2(1 + \sqrt{3}i) + 4$$

   Indeed it works.

Question 6. **Polynomial with a Double Complex Root**

**Question Recap**

$h(x) = (x - 3)(x - (1 + 2i))^2 (x - (1 - 2i))^2$.

1. Expand fully.
2. Explain repeated roots and the theorem.
3. Identify all roots and multiplicities.

**Solution Sketch**

1. **Expansion**
   - First, note $(x - (1 + 2i))(x - (1 - 2i)) = x^2 - 2x + (1 + 4) = x^2 - 2x + 5$.

- Squaring that: $(x^2 - 2x + 5)^2$.
- Multiply the result by $(x - 3)$.

A step-by-step expansion is tedious but straightforward:

$(x^2 - 2x + 5)^2 = x^4 - 4x^3 + 14x^2 - 20x + 25$.

Then

$h(x) = (x - 3)\,(x^4 - 4x^3 + 14x^2 - 20x + 25)$.

Expanding gives a 5th-degree polynomial:

$x^5 - 4x^4 + 14x^3 - 20x^2 + 25x - 3x^4 + 12x^3 - 42x^2 + 60x - 75$

Combine like terms:

$h(x) = x^5 - 7x^4 + 26x^3 - 62x^2 + 85x - 75$.

This is the fully expanded form.

2. **Repeated (Double) Complex Roots**
   - Having $(1 + 2i)$ as a double root means $(x - (1 + 2i))^2$ is a factor. The same goes for $(1 - 2i)$.
   - The Conjugate Root Theorem remains valid: a polynomial with real coefficients must have **both** $1 + 2i$ and $1 - 2i$. The fact they appear with multiplicity 2 does not violate any rule; it simply means each is a repeated factor.
3. **All Roots**
   - $x = 3$ (simple root).
   - $x = 1 + 2i$ (double root).
   - $x = 1 - 2i$ (double root).

**Question 7. Using the Conjugate Root Theorem to Simplify Division**

**Question Recap**
$r(x) = x^4 + 4x^3 + mx^2 + nx + 9$, real $m,n$. A non-real root is $\gamma = -1 + 2i$.

1. Conjugate root?
2. Two linear factors?
3. How to reduce to a quadratic factor?
4. If another real root $x = 1$ were known, how to solve for $m$, $n$?

**Solution Outline**

1. **Conjugate Root**: $\gamma* = -1 - 2i$.
2. **Linear Factors**: $(x - (-1 + 2i)) = (x + 1 - 2i)$ and $(x + 1 + 2i)$. Multiplying them gives a real quadratic factor:

   $(x + 1 - 2i)(x + 1 + 2i) = x^2 + 2x + (1 + 4) = x^2 + 2x + 5$.

3. **Reduce to Quadratic**
   - We know $(x^2 + 2x + 5)$ divides $r(x)$. Divide $r(x)$ by $(x^2 + 2x + 5)$ to get a second-degree quotient $(x^2 + Ax + B)$.
   - Then $r(x) = (x^2 + 2x + 5)(x^2 + Ax + B)$.
4. **Finding *m, n***
   - If another **real root** $x = 1$is known, that root must satisfy the quadratic (1)2 + $A(1) + B = 0$, or we can say $r(1) = 0$.
   - Plugging $x = 1$ into the fully expanded polynomial can give equations in terms of $m, n$.
   - Solve simultaneously with the condition that $(x^2 + 2x + 5)$ is a factor (which also imposes constraints on $m, n$).

   *(In an exam, you might be asked to do the full algebra if all details are given. Here, an outline suffices.)*

**Question 8. Reflective / TOK-Style Extension**

**Question Recap**

- How does the Conjugate Root Theorem connect to the Fundamental Theorem of Algebra?
- Would it still hold in other coefficient systems (not ℝ)?

**Key Talking Points**

- **Connection to FTA**:
  - The Fundamental Theorem of Algebra states that every polynomial of degree $n$ (with complex coefficients) has exactly $n$ roots in ℂ, counting multiplicities.
  - When the coefficients happen to be real, complex (non-real) roots necessarily come in conjugate pairs. This pairing is consistent with FTA's total root count.
- **Different Coefficient Fields**:
  - Over the **complex** coefficients themselves, there's no need for a "conjugate root theorem"; any number can appear as a root without implying anything about its conjugate.
  - Over **finite fields** or other number systems, the usual notion of "conjugation" may not apply the same way, so the statement might fail or require a different interpretation.