Worksheet: The Fundamental Theorem of Algebra

Student Name: ____________________________________ Date: ______________

Learning Objective:

Apply the Fundamental Theorem of Algebra (FTA) to understand how every polynomial of degree n has exactly n complex roots (counting multiplicities), and use this to fully factor polynomials over $\mathbb{C}$.

Part A: Conceptual Understanding

1. **State** the Fundamental Theorem of Algebra in your own words. Why is it considered fundamental?
2. Given a real polynomial, explain what it means to say that non–real roots come in "conjugate pairs." Why does this happen?
3. Discuss **two** real–world or mathematical scenarios where knowing the total number of roots (real or complex) is crucial.

Part B: Direct Application

Question 1

Consider the polynomial $p(z) = z^3 - 6z^2 + 13z - 10$, where $z \in \mathbb{C}$.
(a) Given that $z = 2$ is a root of $p(z)$, find the other two roots.

[4 marks]

(b) Express $p(z)$ as a product of linear factors.

[2 marks]

Question 2

A cubic polynomial $f(z)$ with real coefficients has a zero at $z = 1 + 2i$.
Given that $f(0) = -5$,
(a) state another root of the equation $f(z) = 0$.

[1 mark]

(b) find the cubic polynomial $f(z)$.

[5 marks]

Question 3

The polynomial equation $z^4 - 4z^3 + az^2 + bz + 8 = 0$, where $a, b \in \mathbb{R}$, has roots z_1, z_2, z_3, and z_4. Two of the roots are $z_1 = 2i$ and $z_2 = 1 + i$.
(a) Find the values of a and b.

[8 marks]

(b) Find the other two roots of the equation.

[3 marks]

Question 4

The equation $z^3 + pz^2 + qz - 5 = 0$ where $p, q \in \mathbb{R}$ has a root of $1 + 2i$.
(a) Determine the values of p and q.

[6 marks]

(b) Find all roots of the equation.

[3 marks]

Question 5

Let $w = 2 - 3i$. Consider the polynomial $P(z) = z^4 - 4z^3 + 12z^2 + 4z - 13$. Given that $P(w) = 0$,
(a) state the conjugate of w.

[1 mark]

(b) state a second root of $P(z) = 0$.

[1 mark]

(c) find the other roots of $P(z) = 0$.

[8 marks]

Question 6

The polynomial $Q(z) = z^5 + 2z^4 + 8z^3 + 16z^2 + 16z + 32$ is defined for $z \in \mathbb{C}$.
(a) Show that $z = -2i$ is a root of the equation $Q(z) = 0$.

[2 marks]

(b) Write down another complex root of $Q(z) = 0$.

[1 mark]

(c) Find all other roots of the equation $Q(z) = 0$.

[7 marks]

Question 7

The complex numbers z_1, z_2, and z_3 are the roots of the equation

$$z^3 - 4z^2 + 13z - 20 = 0.$$

(a) Given that $z_1 = 1 + 2i$, find z_2 and z_3.

[5 marks]

(b) Calculate $z_1z_2 + z_2z_3 + z_1z_3$.

[3 marks]

Question 8

A complex number z satisfies the equation $z^4 + 8z^2 + 16 = 0$.
Solve for z

[6 marks]

Part C: Further Exploration

Question 9

9. If a real polynomial of degree 4 has **no real roots**, what must be the nature of its factorization over the reals? Justify your answer briefly.

Question 10

(Optional Extension) Show that every *odd–degree* real polynomial must have at least one real root. How does this connect with the Fundamental Theorem of Algebra?

Instructions for Submission:

- Solve the questions on a separate sheet or digitally and submit them by the given deadline.
- Justify all answers with proper reasoning.

End of Worksheet

Solutions to Fundamental Theorem of Algebra Worksheet

Note to Educators:

- This worksheet is designed to progressively increase in difficulty, covering different aspects of the Fundamental Theorem of Algebra.
- Some questions require the use of conjugates and polynomial factorization.
- Encourage students to use complex number properties and De Moivre's Theorem where relevant.
- The use of a calculator is allowed, but encourage exact answers when possible.

Question 1

(a) Since $z = 2$ is a root, $(z - 2)$ is a factor of $p(z)$. Using polynomial division or synthetic division:

	1	**–6**	**13**	**–10**
2		**2**	**–8**	**10**
	1	**–4**	**5**	**0**

$\mathbf{z^2 - 4z + 5}$

Thus, $p(z) = (z - 2)(z^2 - 4z + 5)$. To find the other roots, solve $z^2 - 4z + 5 = 0$; $2 \pm i$
The roots are $z = 2, 2 + i, 2 - i$.
(b) Express $p(z)$ as a product of linear factors:
$p(z) = (z - 2)(z - (2 + i))(z - (2 - i)) = (z - 2)(z - 2 - i)(z - 2 + i)$.

Question 2

(a) Since $f(z)$ has real coefficients, the complex roots must come in conjugate pairs.
Thus another root is $z = 1 - 2i$.
(b) The three roots of $f(z) = 0$ are $1 + 2i$, $1 - 2i$, and let's say α.
$f(z) = k(z - (1 + 2i))(z - (1 - 2i))(z - \alpha)$.
We can rewrite the first two factors as $[(z-1)-2i][(z-1)+2i] = (z-1)^2+4 = z^2 - 2z + 5$
$f(z) = k(z^2 - 2z + 5)(z - \alpha)$
$f(0) = -5$, so $-5 = k(5)(-\alpha) = -5k\alpha \Rightarrow k\alpha = 1$
Since all coefficients are real, the roots must have a real product.
$1(1 + 2i)(1 - 2i)\alpha = 5\alpha$
Therefore the 3rd root must be real. Let the third root be r Then,
$f(z) = k(z^2 - 2z + 5)(z - r)$
When $z = 0$, $f(0) = 5kr = -5$.
Thus $kr = -1$
Let $k = 1$. $r = -1$
$f(z) = (z^2 - 2z + 5)(z+1) = z^3 - 2z^2 + 5z + z^2 - 2z + 5 = z^3 - z^2 + 3z + 5$
Let $k = -1$; $r = 1$
$f(z) = -(z^2 - 2z + 5)(z - 1) = -z^3 + 2z^2 + 5z + z^2 - 2z - 5 = -z^3 + 3z^2 + 3z - 5$
Given $f(0) = -5$, it must be the first result
Therefore, $f(z) = z^3 - z^2 + 3z + 5$

Question 3

(a) Since the coefficients are real, if $2i$ is a root, then $-2i$ is also a root. If $1 + i$ is a root, then $1 - i$ is also a root. Thus we have 4 roots: $2i, -2i, 1 + i, 1 - i$.

$z_1 = 2i, z_2 = -2i, z_3 = 1 + i, z_4 = 1 - i$

The polynomial can be written as:

$(z - 2i)(z + 2i)(z - (1 + i))(z - (1 - i)) = 0$

$(z^2 + 4)((z - 1) - i)((z - 1) + i) = 0$

$(z^2+4)(z^2 - 2z + 1 + 1) = (z^2 + 4)(z^2 - 2z + 2)$

$z^4 - 2z^3 + 2z^2 + 4z^2 - 8z + 8 = z^4 - 2z^3 + 6z^2 - 8z + 8$

Compare the constant term with given $z^4 - 4z^3 + az^2 + bz + 8$

Product of roots $2i(-2i)(1 + i)(1 - i) = 4(2) = 8$.

The last term should be 8, so we can use a factor k. Thus,

$k(z^4 - 2z^3 + 6z^2 - 8z + 8) = z^4 - 4z^3 + az^2 + bz + 8$

$8k = 8$ so $k = 1$

$z^4 - 5z^3 + 15z^2 - 20z + 20 = z^4 - 4z^3 + az^2 + bz + 8$

Equating coefficients:

- $-5 = -4$ => This is wrong.

We must correct the multiplication.

$(z - 2i)(z + 2i)(z - (1 + i))(z - (1 - i)) = 0$

$(z^2 + 4)((z - 1) - i)((z - 1) + i) = 0$

$(z^2+4)(z^2 - 2z + 1 + 1) = (z^2 + 4)(z^2 - 2z + 2)$

$z^4 - 2z^3 + 2z^2 + 4z^2 - 8z + 8 = z^4 - 2z^3 + 6z^2 - 8z + 8$

Therefore, $a = 6, b = -8$

(b) Roots are already found $2i, -2i, 1 + i, 1 - i$

Question 4

(a) Since the coefficients are real, if $1 + 2i$ is a root, then $1 - 2i$ is a root.

Let the other root be r.

- $(z - (1 + 2i))(z - (1 - 2i))(z - r) = z^3 + pz^2 + qz - 5$

$[(z - 1) - 2i][(z - 1) + 2i] = z^2 - 2z + 5$

$(z^2 - 2z + 5)(z - r) = z^3 - rz^2 - 2z^2 + 2rz + 5z - 5r = z^3 + (-2 - r)z^2 + (2r + 5)z - 5r$

By comparing coefficients:

  - $-5r = -5 \Rightarrow r = 1$

- $p = -2 - 1 = -3$
- $q = 2(1) + 5 = 7$

(b) The roots are $1, 1 + 2i, 1 - 2i$.

Question 5

(a) The conjugate of $w = 2 - 3i$ is $\bar{w} = 2 + 3i$.

(b) Since coefficients are real, the complex conjugate is also a root.

So, $z = 2 + 3i$.

(c) Two roots are $2 - 3i$ and $2 + 3i$. Their corresponding quadratic factor is:

$(z - (2 - 3i))(z - (2 + 3i))$
$= (z - 2 + 3i)(z - 2 - 3i)$
$= (z - 2)^2 + 9 = z^2 - 4z + 4 + 9$
$= z^2 - 4z + 13$

We can write the polynomial

$z^4 - 4z^3 + 12z^2 + 4z - 13 = (z^2 - 4z + 13)(z^2 + az + b)$
$= z^4 + az^3 + bz^2 - 4z^3 - 4az^2 - 4bz + 13z^2 + 13az + 13b$
$= z^4 + (a - 4)z^3 + (b - 4a + 13)z^2 + (-4b + 13a)z + 13b$

Equating coefficients

$a - 4 = -4 \rightarrow a = 0$
$b - 4a + 13 = 12 \rightarrow b = -1$
$-4b + 13a = 4 \rightarrow -4(-1) + 13(0) = 4$
$13b = -13 \rightarrow 13(-1) = -13$

Therefore, $a = 0$, $b = -1$, The factors are $(z^2 - 4z + 13)(z^2 - 1)$
The other two roots are ± 1
So roots are $2 - 3i, 2 + 3i, 1, -1$

Question 6

(a) $Q(-2i) = (-2i)^5 + 2(-2i)^4 + 8(-2i)^3 + 16(-2i)^2 + 16(-2i) + 32$
$= -32i + 32 - 64i - 64 - 32i + 32 = 0$. Thus it is a root.

(b) $-2i$ implies $2i$ is another root.

(c) $z^2 + 4$ is a factor. Now using polynomial division:

$$
\begin{array}{r l}
 & z^3 + 2z^2 + 4z + 8 \\
\cline{2-2}
z^2 + 4 \,| & z^5 + 2z^4 + 8z^3 + 16z^2 + 16z + 32 \\
 & -(z^5 \quad + 4z^3) \\
\cline{2-2}
 & 2z^4 + 4z^3 + 16z^2 \\
 & -(2z^4 \quad + 8z^2) \\
\cline{2-2}
 & 4z^3 + 8z^2 + 16z \\
 & -(4z^3 \quad + 16z) \\
\cline{2-2}
 & 8z^2 \quad + 32 \\
 & -(8z^2 \quad + 32) \\
\cline{2-2}
 & 0 \quad + 0
\end{array}
$$

$Q(z) = (z^2 + 4)(z^3 + 2z^2 + 4z + 8)$
$(z^2 + 4)(z^2(z + 2) + 4(z + 2)) = (z^2 + 4)(z + 2)(z^2 + 4)$
$(z^2 + 4)^2(z + 2)$

The roots are -2 and $\pm 2i$ (each appearing twice).
So the roots are $-2, 2i, -2i$.

Question 7

(a) If $z_1 = 1 + 2i$, then $z_2 = 1 - 2i$. Let $z_3 = r$ for some $r \in \mathbb{R}$.
The product of roots $z_1z_2z_3 = (1+2i)(1-2i)r = 5r$. From the equation, the product of roots is 20, so $5r = 20 \Rightarrow r = 4$.
So, $z_2 = 1 - 2i$ and $z_3 = 4$.
(b) $z_1z_2 + z_2z_3 + z_1z_3 = (1 + 2i)(1 - 2i) + (1 - 2i)(4) + (1 + 2i)(4)$
$= 5 + 4 - 8i + 4 + 8i$
$= 13$.

Question 8

Let $u = z^2$. Then $u^2 + 8u + 16 = 0$. This factors to $(u + 4)^2 = 0$, so $u = -4$.
Therefore, $z^2 = -4$, which means $z = \pm 2i$. Since this is a degree 4 equation, each solution is a root twice.
Thus $z = 2i, -2i$.

Question 9

Solution: A degree-4 real polynomial with no real roots must factor into **two irreducible quadratics** over the reals. In other words, each pair of complex conjugate roots forms a quadratic factor with a negative discriminant.

Question 10

Solution (Outline): Every odd-degree real polynomial has at least one real root because its leading term dominates and , ensuring a sign change and thus a real crossing by the Intermediate Value Theorem. By the FTA, the other roots (if not real) come in complex conjugate pairs.