Lesson Plan: The Fundamental Theorem of Algebra

Subject: Mathematics
Course: IB Mathematics Analysis and Approaches
Level: IB HL
Topic: The Fundamental Theorem of Algebra
Duration: 80 minutes

Learning Objective:

- **Key Goal:** Students will **understand and apply the Fundamental Theorem of Algebra (FTA)** to polynomials, recognizing that any degree-n polynomial has exactly n complex roots (counting multiplicities).
- Students will also learn about the implications for **factorization** over real and complex numbers, including the role of irreducible quadratic factors when coefficients are real.

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1. Lesson Introduction (10 minutes)

Engagement – Inquiry Prompt:

- Display the 4th-degree polynomial: $f(x) = x^4 - 6x^3 + 11x^2 - 6x$ from the slides.
- Ask: *"How many real solutions might it have? Could it have complex solutions?"*
- *"Do we always know how many complex solutions it must have?"*.

Objective connection:

- Introduce the **Fundamental Theorem of Algebra** in informal terms:
  "Every polynomial of degree n has exactly n solutions (roots) in the complex number system, counting multiplicities."

Relevance

- Explain that this theorem **unifies** real and complex solutions: *"Even if we can't factor a polynomial using real numbers alone, the theorem guarantees we can factor it fully in the complex field."*

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2. Exploration – Guided Inquiry (20 minutes)

Activity: Small Group Investigation

- Split students into small groups and give each group a **different polynomial** (various degrees, some easily factorable over reals, others requiring complex factors).
- Task:
  - Predict how many total roots (real or complex) the polynomial should have based on its degree.

- Attempt to find any *obvious* real roots (e.g., using the Factor Theorem or integer-root tests).
- If they get "stuck" with real factorization, encourage them to consider possible *complex* roots and/or use the discriminant for a quadratic factor.

- **Discussion Prompts:**
  - *"What happens if you can't factor the polynomial fully over the reals?"*
  - *"How can we systematically confirm a polynomial's factors if complex numbers are involved?"*

Teacher Circulation

- Move between groups, pose guiding questions, and remind students about the possibility of conjugate roots for polynomials with real coefficients.

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3. Application – Connecting to the PPT (25 minutes)

- **Examples from Presentation:**
  - Refer to a **polynomial** given in the PowerPoint slides (e.g., $f(x) = x^4 - 6x^3 + 11x^2 - 6x$).
  - *"We see it factors as* $x(x-1)(x-2)(x-3)$*. So all four roots are real. But not all degree-4 polynomials factor so nicely in real terms."*
- **Demonstrate Complex Factorization**
  - Show another polynomial from the slides (e.g., $x^3 - 8$ or one that yields a pair of complex conjugate roots) to illustrate how:
    - We find one real factor (e.g., $x - 2$ if 2 is a root),
    - We reduce the polynomial's degree,
    - We factor the remaining quadratic and possibly get complex roots.

Key Observations

- Real polynomials: if they have a non-real root ($a + bi$), then ($a - bi$) is also a root.
- Over the reals, polynomials factor into a product of linear and/or irreducible quadratic factors.

Small Practice

- Have students quickly practice factoring a polynomial known to have complex solutions (e.g., $x^2 + x + 1$ or a cubic from the slides).
- Emphasize the connection to the theorem: *"Even though the roots aren't real, they still count toward the total of n complex roots."*

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4. Reflection and Consolidation (15 minutes)

Student Reflection Questions:

- *"How does knowing that every polynomial of degree n has n complex roots guide our factorization strategies?"*
- *"Why do you think real polynomials always come with complex conjugate pairs if they have non-real solutions?"*

Exit Ticket:

- Provide a short polynomial (e.g., $P(x) = x^5 - 4x^4 - 3x^3 + 34x^2 - 52x + 24$, from the slides) and ask:
  - *"Using the Fundamental Theorem of Algebra as a guide, how many roots should you expect in total (counting multiplicities)?"*
  - *"Find at least one root and comment on whether it might appear multiple times."*

Assessment & Homework

- **Formative Assessment:**
  - Observe student discussions, check their group factorization attempts, and clarify misconceptions..
- **Homework:**
- Pick 1–2 polynomials from the slides or a textbook with known complex roots and **fully factor** them (over the complexes).
- Write a short paragraph about how the Fundamental Theorem of Algebra ensures a complete factorization.

Differentiation

- **For Advanced Students**:
  - Introduce the **complex plane** more formally and discuss how non-real roots appear in conjugate pairs for polynomials with real coefficients.
  - Invite them to explore **why** the theorem holds (brief historical perspective or outline of a proof approach using complex analysis).
- **For Struggling Students**:
  - Emphasize the **Remainder Theorem and Factor Theorem** connections, as well as basic synthetic division, before bridging to the fundamental theorem.
  - Provide more direct scaffolding for factoring quadratics with negative discriminants.