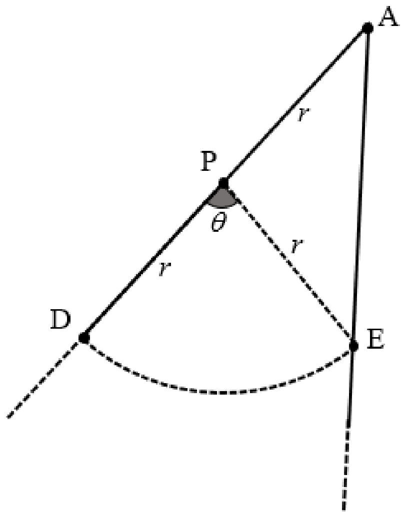
1.6.1 Simple deductive proof

Student name: ___________________________________ Score: _______

1. Two straight fences meet at Point A and a field lies between them.

A horse is tied to a Post, P, by a rope of length r metres. Point D is on one fence and point E is on the other, such that PD = PE = PA = r and $D\hat{P}E = \theta$ radians. This is shown in the following diagram.



The length of the arc DE shown in the diagram is 32 m.

(a) Write down an expression for r in terms of θ.

(b) Show that the area of the field that the horse can reach is $\frac{512}{\theta^2}(\theta + \sin\theta)$.

2. Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = e^{x-2} + k$ where $h,\ k \in \mathbb{R}$.

(a) Find $f'(x)$.

The graphs of f and g have a common tangent at $x = 3$.

(b) Show that $h = \dfrac{e+6}{2}$.

(c) Hence, show that $k = e + \dfrac{e^2}{4}$.

3. Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$.

4. Let $y = \dfrac{\ln x}{x^4}$ for $x > 0$.

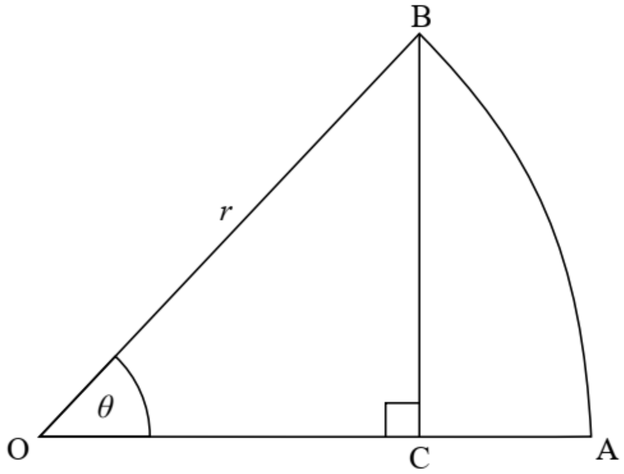
Show that $\dfrac{dy}{dx} = \dfrac{1 - 4\ln x}{x^5}$.

5. Consider two consecutive positive integers, n and $n + 1$.

Show that the difference of their squares is equal to the sum of the two integers.

6. OAB is a sector of the circle with centre O and radius r, as shown in the following diagram.

diagram not to scale



The angle AOB is θ radians, where $0 < \theta < \frac{\pi}{2}$.

The point C lies on OA and OA is perpendicular to BC.

Show that OC = $r \cos \theta$.

7. All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by

$A = A_0 e^{-kt}$ where $t \geq 0$ and A_0, k are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$

8. It is given that $\log_{ab} a = 3$, where $a, b \in \mathbb{R}^+$, $ab \neq 1$.

Show that $\log_{ab} b = -2$.

9. Consider the arithmetic sequence $\log_8 27$, $\log_8 p$, $\log_8 q$, $\log_8 125$, where $p > 1$ and $q > 1$.

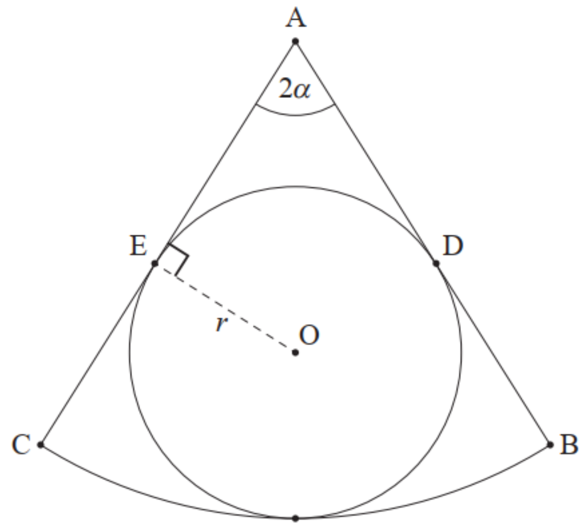
Show that 27, p, q and 125 are four consecutive terms in a geometric sequence.

10. The following diagram shows a sector ABC of a circle with centre A. The angle $\mathrm{B\hat{A}C} = 2\alpha$, where $0 < \alpha < \frac{\pi}{2}$, and $\mathrm{O\hat{E}A} = \frac{\pi}{2}$.

A circle with centre O and radius r is inscribed in sector ABC.

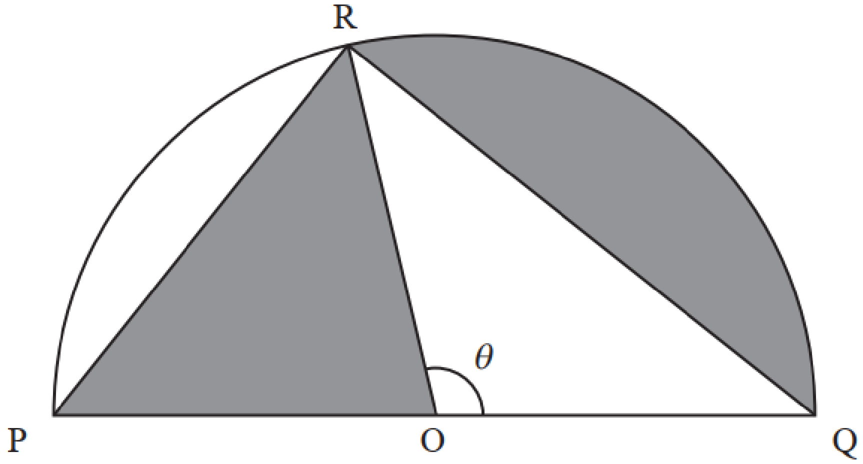
AB and AC are both tangent to the circle at points D and E respectively.

diagram not to scale



Show that the area of the quadrilateral ADOE is $\dfrac{r^2}{\tan\alpha}$.

11. The following diagram shows a semicircle with centre O and radius r. Points P, Q and R lie on the circumference of the circle, such that $\mathrm{PQ} = 2r$ and $\mathrm{R\hat{O}Q} = \theta$, where $0 < \theta < \pi$.



Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$.

12. Let a be a constant, where $a > 1$.

Show that $a^2 + \left(\frac{a^2 - 1}{2}\right)^2 = \left(\frac{a^2 + 1}{2}\right)^2$.