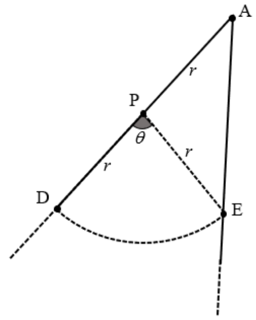
1.6.1 Simple deductive proof

Student name: ANSWERS Score: _______

1. Two straight fences meet at Point A and a field lies between them.

A horse is tied to a Post, P, by a rope of length r metres. Point D is on one fence and point E is on the other, such that PD = PE = PA = r and $D\hat{P}E = \theta$ radians. This is shown in the following diagram.



The length of the arc DE shown in the diagram is 32 m.

(a) Write down an expression for r in terms of θ. $r = \frac{32}{\theta}$

(b) Show that the area of the field that the horse can reach is $\frac{512}{\theta^2}(\theta + \sin\theta)$.

$\frac{1}{2}r^2\theta + \frac{1}{2}r^2\sin(\pi - \theta) = \frac{512}{\theta^2}(\theta + \sin\theta)$

$r = \frac{32}{\theta} \Rightarrow$ $\frac{1}{2}\left(\frac{32}{\theta}\right)^2\theta + \frac{1}{2}\left(\frac{32}{\theta}\right)^2\sin(\pi - \theta)$

$\sin(\pi - \theta) = \sin\theta$ $\frac{1}{2}\left(\frac{32}{\theta}\right)^2\theta + \frac{1}{2}\left(\frac{32}{\theta}\right)^2\sin\theta$

$\frac{32^2}{2\theta^2}(\theta + \sin\theta) = \frac{512}{\theta^2}(\theta + \sin\theta)$

2. Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = e^{x-2} + k$ where $h, k \in \mathbb{R}$.

(a) Find $f'(x)$. $f'(x) = -2(x - h)$

The graphs of f and g have a common tangent at $x = 3$. $g'(x) = e^{x-2}$

(b) Show that $h = \frac{e+6}{2}$.

$f'(3) = g'(3)$
$-2(3 - h) = e^{3-2}$
$-6 + 2h = e$

(c) Hence, show that $k = e + \frac{e^2}{4}$.

$f(3) = g(3)$
$-(3 - h)2 + 2k = e^{3-2} + k$
$-\left(3 - \frac{e+6}{2}\right)^2 + 2k = e + k$
$k = e + \left(\frac{6+e-6}{2}\right)^2$

3. Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$.

Attempt to use double angle formula for $\sin 2x$ or $\cos 2x$
LHS=$2\sin x\cos x + \cos 2x - 1$ or $\sin 2x + 1 - 2\sin^2 x - 1$ or $2\sin x\cos x + 1 - 2\sin^2 x - 1 = 2\sin x\cos x - 2\sin^2 x$
$\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$ = RHS

4. Let $y = \frac{\ln x}{x^4}$ for $x > 0$.

Show that $\frac{dy}{dx} = \frac{1 - 4\ln x}{x^5}$.

Using the quotient rule $\frac{dy}{dx} = \frac{x^4\left(\frac{1}{x}\right) - (\ln x)(4x^3)}{(x^4)^2}$

$= \frac{x^3 - 4x^3 \ln x}{x^8}$

$= \frac{x^3(1 - 4\ln x)}{x^8}$

Simplifying $= \frac{1 - 4\ln x}{x^5}$

5. Consider two consecutive positive integers, n and $n + 1$.

Show that the difference of their squares is equal to the sum of the two integers.

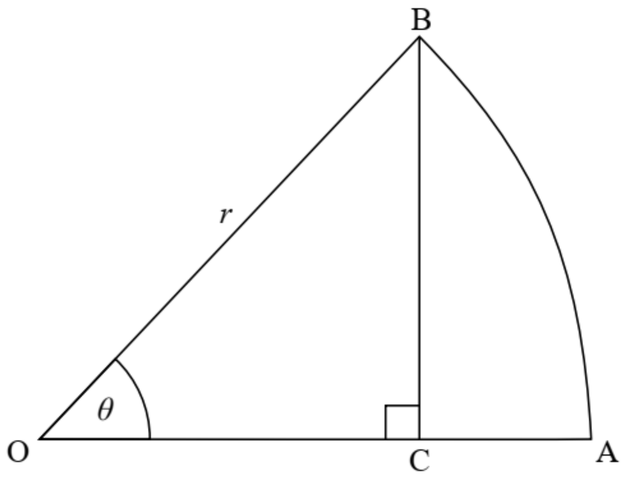
$(n + 1)^2 - n^2 = 2n + 1$

$n^2 + 2n + 1 - n^2 = 2n + 1$

$2n + 1 = 2n + 1$

6. OAB is a sector of the circle with centre O and radius r, as shown in the following diagram.

diagram not to scale



The angle AOB is θ radians, where $0 < \theta < \frac{\pi}{2}$.

The point C lies on OA and OA is perpendicular to BC.

Show that OC = $r \cos \theta$.

$\cos \theta = \frac{OC}{r}$

$OC = r \cos \theta$

7. All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by

$A = A_0 e^{-kt}$ where $t \geq 0$ and A_0, k are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$

$100 = A_0 e^{-k(0)}$

$100 = A_0$

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$

$50 = 100 e^{-k(5730)}$

$\frac{1}{2} = e^{-5730k}$

$2 = e^{5730k}$

$\ln 2 = 5730k$

$\frac{\ln 2}{5730} = k$

8. It is given that $\log_{ab} a = 3$, where $a, b \in \mathbb{R}^+$, $ab \neq 1$.

Show that $\log_{ab} b = -2$.

METHOD 1

$\log_{ab} ab = 1$

$\log_{ab} a + \log_{ab} b = 1$

$3 + \log_{ab} b = 1$

$\log_{ab} b = -2$

METHOD 2

$(ab)^3 = a$

$b = \frac{a}{a^3b^2}$ or $b^3 = a^{-2}$ or $b^{-1} = (ab)^2$ or $b^3 = \frac{1}{a^2}$

$b = \frac{1}{a^2b^2}$ or $b = (ab)^{-2}$ or $3\log_{ab} b = -2\log_{ab} a$

or $-\log_{ab} b = 2\log_{ab} ab$

$\log_{ab} b = -2$

METHOD 3

$(ab)^3 = a$

$\log_{ab} (ab)^3 = \log_{ab} a$ or $\log_{ab} a^3b^3 = \log_{ab} a$

$3\log_{ab} a + 3\log_{ab} b = \log_{ab} a$

$3\log_{ab} b = -2\log_{ab} a$ or $\log_{ab} b^3 = \log_{ab} a^{-2}$

$\log_{ab} b = \frac{-2}{3}\log_{ab} a$ or $\log_{ab} b = \frac{-2}{3}(3)$ or $\log_{ab} b = \log_{ab} a^{-\frac{2}{3}}$

$\log_{ab} b = -2$

METHOD 4

$\log_{ab} a = 3$

Writing in terms of base a

$\frac{\log_a a}{\log_a ab} = 3$

$\frac{\log_a a}{\log_a a + \log_a b} = 3$ or $\frac{1}{1 + \log_a b} = 3$

$3 + 3\log_a b = 1$ or $3\log_a b = -2$

or $\log_a b = \frac{-2}{3}$

Writing $\log_{ab} b$ in terms of base a

$\log_{ab} b = \frac{\log_a b}{\log_a a + \log_a b} = \frac{\frac{-2}{3}}{1 - \frac{2}{3}}$ or $\frac{\frac{-2}{3}}{\frac{1}{3}}$

$\log_{ab} b = -2$

9. Consider the arithmetic sequence $\log_8 27$, $\log_8 p$, $\log_8 q$, $\log_8 125$, where $p > 1$ and $q > 1$.

Show that 27, p, q and 125 are four consecutive terms in a geometric sequence.

METHOD 1

$u_2 - u_1 = u_4 - u_3$ or $(u_3 - u_2)$

$\log_8 p - \log_8 27 = \log_8 125 - \log_8 q$

$\log_8 \left(\frac{p}{27}\right) = \log_8 \left(\frac{125}{q}\right)$

$\frac{p}{27} = \frac{125}{q}$

METHOD 2

$p = 27(8^d)$ and $q = 27(8^{2d})$ or $q = 27(8^{2d})$ and $125 = 27(8^{3d})$

two correct pairs of consecutive terms, in terms of d

$\frac{27(8^d)}{27} = \frac{27(8^{2d})}{27(8^d)} = \frac{27(8^{3d})}{27(8^{2d})}$ all simplify to 8^d

$27, p, q$ and 125 are in geometric sequence

10. The following diagram shows a sector ABC of a circle with centre A. The angle $\mathrm{B\hat{A}C} = 2\alpha$, where $0 < \alpha < \frac{\pi}{2}$, and $\mathrm{O\hat{E}A} = \frac{\pi}{2}$.

A circle with centre O and radius r is inscribed in sector ABC.

AB and AC are both tangent to the circle at points D and E respectively.

diagram not to scale

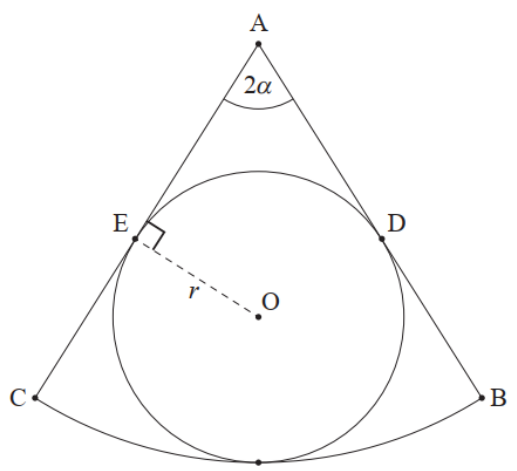
$\tan \alpha = \frac{r}{AE}$

$AE = \frac{r}{\tan \alpha}$

Area of ADOE is 2 x area of triangle AOE

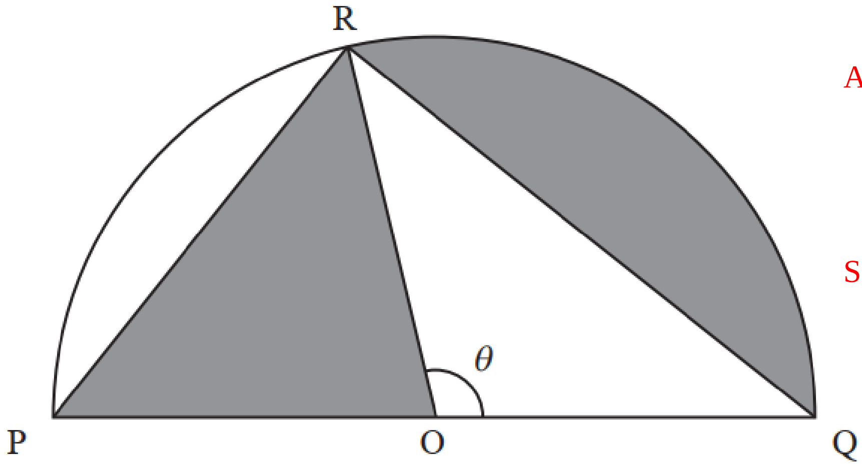
Area AOE $= \frac{1}{2} r \times AE$ or $\frac{1}{2} r \times \frac{r}{\tan \alpha}$ or $\frac{r^2}{2\tan \alpha}$

Area ADOE $= 2 \times \frac{r^2}{2\tan \alpha} = \frac{r^2}{\tan \alpha}$



Show that the area of the quadrilateral ADOE is $\frac{r^2}{\tan \alpha}$.

11. The following diagram shows a semicircle with centre O and radius r. Points P, Q and R lie on the circumference of the circle, such that $PQ = 2r$ and $R\hat{O}Q = \theta$, where $0 < \theta < \pi$.



Area of the segment = Area of the triangle

$$\tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta = \tfrac{1}{2}r^2\sin(\pi-\theta)$$

$$\tfrac{1}{2}r^2(\theta - \sin\theta) = \tfrac{1}{2}r^2\sin(\pi-\theta)$$

$$(\theta - \sin\theta) = \sin(\pi-\theta)$$

Since $\sin(\pi-\theta) = \sin\theta$

$$\theta - \sin\theta = \sin\theta$$

$$\theta = 2\sin\theta$$

Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$.

12. Let a be a constant, where $a > 1$.

Show that $a^2 + \left(\frac{a^2-1}{2}\right)^2 = \left(\frac{a^2+1}{2}\right)^2$.

METHOD 1 (working with LHS)

Expanding $a^2 + \frac{a^4-2a^2+1}{4}$

$$= \frac{4a^2+a^4-2a^2+1}{4}$$

$$= \frac{a^4+2a^2+1}{4}$$

Factorising

$$= \left(\frac{a^2+1}{2}\right)^2 = \text{RHS}$$

METHOD 2 (working with RHS)

Expanding $\frac{a^4+2a^2+1}{4}$

Changing $2a^2$ by $4a^2 - 2a^2$

$$= \frac{4a^2+a^4-2a+1}{4}$$

$$= a^2 + \frac{a^4-2a^2+1}{4}$$

$$= a^2 + \left(\frac{a^2-1}{2}\right)^2 = \text{LHS}$$