Proof by Exhaustion

Name: ______________________________________ Score: _______

Question 1: Prove that for all integers n, $n^2 + n$ is always even.

Hint:

- Consider two cases: when n is even and when n is odd.

Question 2: Prove that a number is divisible by 5 if and only if its last digit is either 0 or 5.

Hint:

- Split the problem into cases based on the possible last digits of a number.

Question 3: Prove that the sum of the squares of any two consecutive integers is always odd.

Hint:

- Let the two consecutive integers be n and $n + 1$ and consider separate cases for when n is even and when n is odd.

Question 4: Prove that the product of any three consecutive integers is divisible by 6.

Hint:

- Analyse the product of n, $n + 1$, and $n + 2$ in cases when n is divisible by 2 or 3.

Question 5: Prove that for any integer n, $n^3 - n$ is divisible by 6.

Hint:

- Factor the expression $n^3 - n$, then analyse the resulting product.

Question 6: Prove that for all integers n, $n^2 + 3n$ is divisible by 2.

Hint:

- Consider two cases: when n is even and when n is odd.

Question 7: Prove that a number is divisible by 4 if and only if its last two digits are divisible by 4.

Hint:

- Express the number as $n = 100k + bn = 100k + bn = 100k + b$, where b represents the last two digits.

Question 8: Prove that the product of any four consecutive integers is divisible by 24.

Hint:

- Factor the product $n(n + 1)(n + 2)(n + 3)$ and analyze cases when n is divisible by 2 or 3.

Question 9: Prove that if n is a prime number greater than 3, then $n^2 - 1$ is divisible by 24.

Hint:

- Factor $n^2 - 1$ as $(n - 1)(n + 1)$, and analyse the properties of three consecutive integers.

Question 10: Prove that for any positive integer n, $n^5 - n$ is divisible by 5.

Hint:

- Factor the expression $n^5 - n$ and analyze the resulting product.

Solutions

Solution 1: We are asked to prove that $n^2 + n$ is always even.

Step 1: Consider the case when *n* is even.

- Let $n = 2k$, where k is an integer.
- Then $n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$, which is even.

Step 2: Consider the case when *n* is odd.

- Let $n = 2k + 1$, where k is an integer.
- Then $n^2 + n = (2k + 1)^2 + (2k + 1) = 4k^2 + 4k + 1 + 2k + 1 = 2(2k^2 + 3k + 1)$, which is even.

Thus, $n^2 + n$ is always even.

Solution 2:

We are asked to prove that a number is divisible by 5 if and only if its last digit is either 0 or 5.

Step 1: Consider the last digit of the number.

- A number n can be written as $n = 10k + d$, where d is the last digit.
- n is divisible by 5 if $10k + d \equiv 0 \bmod 5$.

Step 2: Analyze the possible values for *d* (the last digit).

- The possible values of d are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
- Only when $d = 0$ or $d = 5$, $10k + d \equiv 0 \bmod 5$.

Thus, a number is divisible by 5 if and only if its last digit is 0 or 5.

Solution 3:

We are asked to prove that the sum of the squares of two consecutive integers is always odd.

Step 1: Consider the case when nnn is even.

- Let $n = 2k$, where k is an integer.
- Then the consecutive integers are $n = 2k$ and $n + 1 = 2k + 1$.
- Their squares are $n^2 = 4k^2$ and $(n + 1)^2 = 4k^2 + 4k + 1$.
- Therefore, $n^2 + (n+1)^2 = 4k^2 + 4k^2 + 4k + 1 = 8k^2 + 4k + 1$, which is odd.

Step 2: Consider the case when n is odd.

- Let $n = 2k + 1$, where k is an integer.
- Then the consecutive integers are $n = 2k + 1$ and $n + 1 = 2k + 2$.
- Their squares are $n^2 = 4k^2 + 4k + 1$ and $(n + 1)^2 = 4k^2 + 8k + 4$.
- Therefore, $n^2 + (n+1)^2 = 4k^2 + 4k + 1 + 4k^2 + 8k + 4 = 8k^2 + 12k + 5$, which is odd.

Thus, the sum of the squares of two consecutive integers is always odd.

Solution 4:

We are asked to prove that the product of any three consecutive integers is divisible by 6.

Step 1: Consider the product $n(n + 1)(n + 2)$.

- Among three consecutive integers, one is divisible by 2 and one is divisible by 3.

Step 2: Analyze two cases:

- **Case 1:** n is divisible by 2**.**
  - Then n is divisible by 2, and among n, $n + 1$, and $n + 2$, one must be divisible by 3.
- **Case 2: n is not divisible by 2.**
  - If n is odd, then $n +$ is divisible by 2, and again one of the three integers is divisible by 3.

Thus, the product of three consecutive integers is always divisible by 6.

Solution 5:

We are asked to prove that for any integer n, n3 – n is divisible by 6.

Step 1: Factor $n^3 - n$.

- We can factor the expression as $n^3 - n = n(n - 1)(n + 1)$, which is the product of three consecutive integers.

Step 2: Prove divisibility by 6.

- As shown in Solution 4, the product of any three consecutive integers is divisible by 6.

Thus, $n^3 - n$ is divisible by 6 for any integer n.

Solution 6:

We are asked to prove that for all integers n, $n^2 + 3n$ is divisible by 2.

Step 1: Consider the case when *n* is even.

- Let $n = 2k$, where k is an integer.
- Then $n^2 + 3n = (2k)^2 + 3(2k) = 4k^2 + 6k = 2(2k^2 + 3k)$, which is divisible by 2.

Step 2: Consider the case when *n* is odd.

- Let $n = 2k + 1$, where k is an integer.
- Then $n^2 + 3n = (2k + 1)^2 + 3(2k + 1) = 4k^2 + 4k + 1 + 6k + 3 = 2(2k^2 + 5k + 2)$, which is divisible by 2.

Thus, $n^2 + 3n$ is divisible by 2 for all integers n.

Solution 7:

We are asked to prove that a number is divisible by 4 if and only if its last two digits are divisible by 4.

Step 1: Express the number as $n = 100k + b$, where b represents the last two digits.

- $n \equiv b \bmod 4$ since $100k \equiv 0 \bmod 4$.
- So, the divisibility of n by 4 depends only on b.

Step 2: Analyze the possible values of b.

- If b is divisible by 4, then $n = 100k + b$ is divisible by 4.
- Conversely, if n is divisible by 4, b must also be divisible by 4 since $n \equiv b \bmod 4$.

Thus, a number is divisible by 4 if and only if its last two digits are divisible by 4.

Solution 8:

We are asked to prove that the product of any four consecutive integers is divisible by 24.

Step 1: Let the four consecutive integers be n, $n + 1$, $n + 2$, and $n + 3$.

Step 2: Consider cases based on $n \bmod 2$ and $n \bmod 3$:

- Among the four consecutive integers, one must be divisible by 2, one must be divisible by 4, and one must be divisible by 3.
- Therefore, the product $n(n + 1)(n + 2)(n + 3)$ is divisible by both 4 and 6, and hence divisible by 24.

Thus, the product of any four consecutive integers is divisible by 24.

Solution 9:

We are asked to prove that if n is a prime number greater than 3, then $n^2 - 1$ is divisible by 24.

Step 1: Factor $n^2 - 1$ as $(n-1)(n+1)$.

- This is the product of two consecutive even numbers.

Step 2: Analyze the properties of three consecutive integers $n - 1$, n, and $n + 1$.

- One of n – 1 or n + 1 is divisible by 4, and one of them is divisible by 2.
- Since n is prime and greater than 3, n is odd, so $n - 1$ and $n + 1$ are both divisible by 2.
- Additionally, one of these numbers is divisible by 3.

Thus, $n^2 - 1$ is divisible by 24 for any prime number $n > 3$.

Solution 10:

We are asked to prove that for any positive integer n, $n^5 - n$ is divisible by 5.

Step 1: Factor $n^5 - n$.

- We can factor $n^5 - n = n(n^4 - 1) = n(n - 1)(n + 1)(n^2 + 1)$.

Step 2: Analyze the product.

- Among n, $n - 1$, and $n + 1$, one of these integers must be divisible by 5.

Thus, $n^5 - n$ is divisible by 5 for any positive integer n.