Worksheet: Proof by Equivalence

Name: ______________________________________ **Date:** __________________

Question 1: Prove that a number n is divisible by 4 if and only if the last two digits of n are divisible by 4.

Hint:

- Consider representing the number n as $100a + b$, where a is the part without the last two digits and b represents the last two digits.

Question 2: Prove that an integer n is odd if and only if n^2 is odd.

Hint:

- Prove both directions:
    - n is odd implies n^2 is odd.
    - n^2 is odd implies n is odd.

Question 3: Prove that $a^2 - b^2 = (a - b)\,(a + b)$ if and only if $a^2 - b^2 = 0$ implies $a = b$.

Hint:

- Use the factorization $a^2 - b^2 = (a - b)\,(a + b)$ to show both directions.

Question 4: Prove that the sum of the squares of two consecutive integers is always odd if and only if the smaller integer is odd.

Hint:

- Represent the two consecutive integers as n and n + 1 and prove both directions.

Question 5: Prove that for any real number x, $x^2 + 2x + 1 = 0$ if and only if $x = -1$.

Hint:

- Factor the quadratic and solve both directions.

Question 6: Prove that for any integer n, $n(n + 1)$ is even if and only if $n(n + 1)\,(n + 2)$ is divisible by 3.

Hint:

- Factor the expressions and consider the properties of consecutive integers.

Solutions

Solution 1:

Let $n = 100a + b$, where b represents the last two digits of n.

Step 1: Prove n divisible by 4 $\implies$ last two digits divisible by 4.

- $n = 100a + b$, and $100 \equiv 0 \bmod 4$. Therefore, $n \equiv b \bmod 4$.
- If n is divisible by 4, then b must be divisible by 4.

Step 2: Prove last two digits divisible by 4 $\implies$ n divisible by 4.

- If b is divisible by 4, then $n = 100a + b \equiv b \equiv 0 \bmod 4$, so n is divisible by 4.

Solution 2:

Let n be an integer.

Step 1: Prove n is odd $\implies n^2$ is odd.

- If n is odd, $n = 2k + 1$ for some integer k.
- Then $n2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd.

Step 2: Prove n^2 is odd $\implies n$ is odd.

- Suppose n^2 is odd. If n were even, say $n = 2\text{k}$, then $n^2 = (2k)^2 = 4k^2$, which is even. Hence, n must be odd.

Solution 3:

We know that $a^2 - b^2 = (a - b)\,(a + b)$.

Step 1: Prove $a^2 - b^2 = 0 \implies a = b$.

- If $a^2 - b^2 = 0$, then $(a - b)\,(a + b) = 0$.
- Therefore, either $a - b = 0$ or $a + b = 0$. The first gives $a = b$, and the second gives $a = -b$, which is not possible unless $a = b$.

Step 2: Prove $a = b \implies a^2 - b^2 = 0$.

- If $a = b$, then $a^2 - b^2 = a^2 - a^2 = 0$.

Solution 4:

Let the two consecutive integers be n and $n + 1$.

Step 1: Prove sum of squares is odd $\Longrightarrow$ smaller integer is odd.

- The sum of squares is $n^2 + (n + 1)^2 = 2n^2 + 2n + 1$.
- For the sum to be odd, $2n^2 + 2n$ must be even, meaning $n(n + 1)$ is even. Thus, n must be odd.

Step 2: Prove smaller integer is odd $\Longrightarrow$ sum of squares is odd.

- If n is odd, then n^2 is odd and $(n + 1)^2$ is even. Therefore, $n^2 + (n + 1)^2$ is odd.

Solution 5:

We are given the equation $x^2 + 2x + 1 = 0$.

Step 1: Prove $x = -1 \Longrightarrow x^2 + 2x + 1 = 0$.

- If x = −1, then (−1)2 + 2(−1) + 1 = 1 −2 + 1 = 0.

Step 2: Prove $x^2 + 2x + 1 = 0 \Longrightarrow x = -1$.

- Factor the quadratic: $x^2 + 2x + 1 = (x + 1)^2 = 0$.
- Therefore, $x = -1$.

Solution 6:

Let n be an integer.

Step 1: Prove n(n + 1) is even $\Longrightarrow$ n(n + 1) (n + 2) divisible by 3.

- If $n(n + 1)$ is even, then one of the numbers is divisible by 2.
  In the product $n(n + 1)\ (n + 2)$, one of the factors is divisible by 3, so the product is divisible by 6.

Step 2: Prove $n(n + 1)\ (n + 2)$ divisible by 3 $\Longrightarrow n(n + 1)$ is even.

- If n(n+1)(n+2)n(n+1)(n+2)n(n+1)(n+2) is divisible by 3, then n(n+1)n(n+1)n(n+1) is divisible by 2, so the product is even.