# Proof by Induction

**Question 1:** Prove by induction that the sum of the first n odd numbers is $n^2$.

$$1 + 3 + 5 + \cdot \cdot \cdot + (2n - 1) = n^2$$

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**Question 2:** Prove by induction that for any integer $n \geq 1$,

$$1^3 + 2^3 + 3^3 + \cdot \cdot \cdot + n^3 = \left(\frac{(n(n+1)}{2}\right)^2$$

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**Question 3:** Prove by induction that for all integers $n \geq 1$,

$$2^n > n$$

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**Question 4:** Prove by induction that for all integers $n \geq 4$,

$$n! > 2^n$$

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**Question 5:** Prove by induction that for any integer $n \geq 1$,

$$1 + 4 + 7 + \cdot \cdot \cdot + (3n - 2) = \frac{n(3n-1)}{2}$$

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**Question 6:** Prove by induction that for all $n \geq 1$,

$5^{n-1}$ is divisible by 4.

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## **Solution 1:** Prove that the sum of the first $n$ odd numbers is $n^2$.

**Step 1: Base Case (n = 1):**

$$1 = 1^2$$

So the base case holds.

**Step 2: Inductive Hypothesis:**

Assume that the formula holds for $n = k$, i.e.,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

**Step 3: Inductive Step:**

We must prove the formula for $n = k + 1$. The sum of the first $k + 1$ odd numbers is:

$$1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1)$$

Using the inductive hypothesis, the sum becomes:

$$k^2 + (2k + 1)$$

Simplifying:

$$k^2 + 2k + 1 = (k + 1)^2$$

Thus, the formula holds for $n = k + 1$.

**Conclusion:** By induction, the formula holds for all $n \geq 1$.

## **Solution 2:** Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

**Step 1: Base Case (n = 1):**

$$1^3 = \left(\frac{1(1 + 1)}{2}\right)^2 = 1$$

So the base case holds.

**Step 2: Inductive Hypothesis:**

Assume the formula holds for $n = k$, i.e.,

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

**Step 3: Inductive Step:**

We need to prove the formula for $n = k + 1$. The sum of the first $k + 1$ cubes is:

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3$$

By the inductive hypothesis, this becomes:

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

Simplifying this expression:

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Thus, the formula holds for $n = k + 1$.

**Conclusion:** By induction, the formula holds for all $n \geq 1$.

## Solution 3: Prove that $2^n > n$ for all $n \geq 1$.

**Step 1: Base Case (n = 1):**

$$2^1 = 2 > 1$$

So the base case holds.

**Step 2: Inductive Hypothesis:**

Assume that $2^k > k$ for some $k \geq 1$.

**Step 3: Inductive Step:**

We need to prove that $2^{k+1} > k + 1$. Using the inductive hypothesis, we know that:

$$2^{k+1} = 2 \times 2^k > 2k$$

Since $2k > k + 1$ for all $k \geq 1$, we have:

$$2^{k+1} > k + 1$$

Thus, the inequality holds for $n = k + 1$.

**Conclusion**: By induction, $2^n > n$ for all $n \geq 1$.

## Solution 4: Prove that $n! > 2^n$ for $n \geq 4$.

**Step 1: Base Case (n = 4):**

$$4! = 24 > 2^4 = 16$$

So the base case holds.

**Step 2: Inductive Hypothesis:**
Assume that $k! > 2^k$ for some $k \geq 4$.

**Step 3: Inductive Step:**
We need to prove that $(k+1)! > 2^{k+1}$. Using the inductive hypothesis:

$$(k+1)! = (k+1)k! > (k+1)2^k$$

Since $k + 1 > 2$ for $k \geq 4$, we have:

$$(k+1)2^k > 2^{k+1}$$

Thus, $(k+1)! > 2^{k+1}$.

**Conclusion**: By induction, $n! > 2^n$ for all $n \geq 4$.

## Solution 5: Prove that $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$.

**Step 1: Base Case (n = 1):**

$$1 = \frac{1(3(1) - 1)}{2} = 1$$

So the base case holds.

**Step 2: Inductive Hypothesis:**
Assume the formula holds for $n = k$, i.e.,

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{k(3k-1)}{2}$$

**Step 3: Inductive Step:**

We need to prove the formula for $n = k + 1$. The sum of the first $k + 1$ terms is:

$$1 + 4 + 7 + \cdots + (3k - 2) + (3(k + 1) - 2)$$

Using the inductive hypothesis:

$$\frac{k(3k - 1)}{2} + (3k + 1) = \frac{(k + 1)(3(k + 1) - 1)}{2}$$

Thus, the formula holds for $n = k + 1$.

**Conclusion:** By induction, the formula holds for all $n \geq 1$.

## Solution 6: Prove that $5^n - 1$ is divisible by 4 for all $n \geq 1$.

**Step 1: Base Case (n = 1):**

$$5^1 - 1 = 4$$

Since 4 is divisible by 4, the base case holds.

**Step 2: Inductive Hypothesis:**

Assume that $5^k - 1$ is divisible by 4 for some $k \geq 1$, i.e.,

$$5^k - 1 = 4m \text{ for some integer } m$$

**Step 3: Inductive Step:**

We need to prove that $5^{k+1} - 1$ is divisible by 4. Using the inductive hypothesis:

$$5^{k+1} - 1 = 5 \times 5^k - 1 = 5(5^k - 1) + 5 - 1$$

By the inductive hypothesis, $5^k - 1$ is divisible by 4, so we can write $5^k - 1 = 4m$ for some integer $m$. Substituting this into the equation:

$$5^{k+1} - 1 = 5 \times 4m + 4 = 20m + 4$$

Since $20m + 4$ is clearly divisible by 4, we have that $5^{k+1} - 1$ is divisible by 4.

### Conclusion:

By induction, $5^n - 1$ is divisible by 4 for all $n \geq 1$.